Further Generalization of Faustmann's Formula for Stochastic Interest Rates

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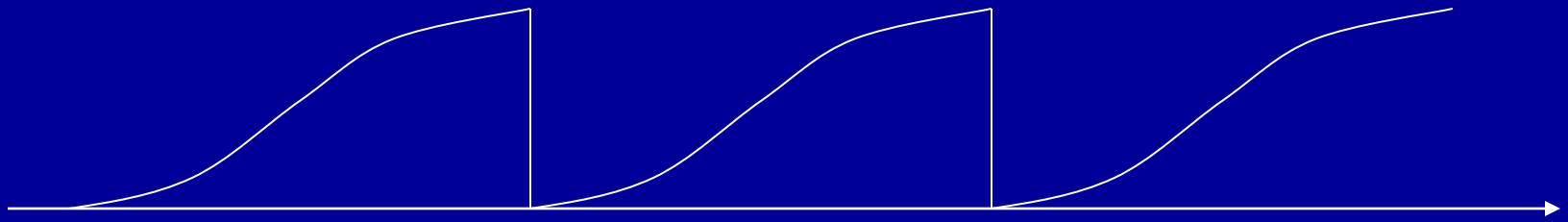
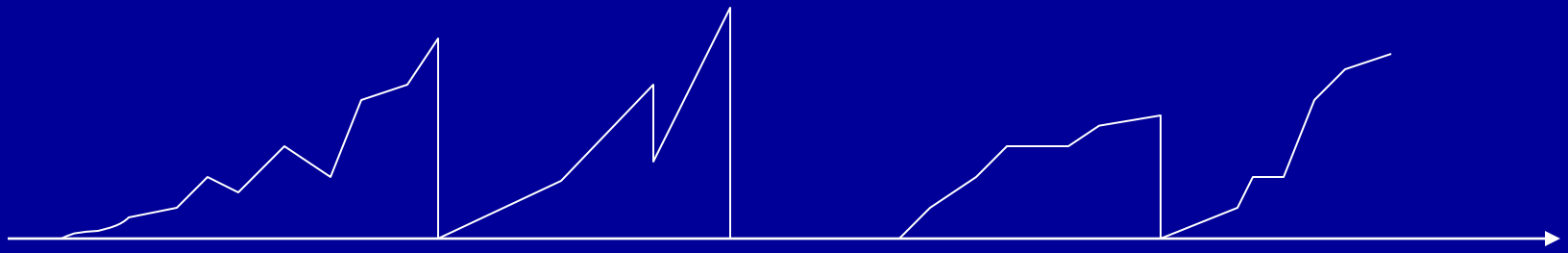
FAUSTMANN MODEL



REALITY



Faustmann-Markov equivalent

	Do nothing						Harvest & plant	
State	0	1	2	3	4	5	1	$/ha
0	1.0						1.0	-494
1			1.0				1.0	-117
2				1.0			1.0	3068
3					1.0		1.0	6396
4						1.0	1.0	8970
5						1.0	1.0	10790

Stochastic growth

	Do nothing						Harvest & plant	
State	0	1	2	3	4	5	0	1
0	1.0						0.1	0.9
1	0.1	0.1	0.7	0.1			0.1	0.9
2	0.1		0.1	0.7	0.1		0.1	0.9
3	0.1			0.1	0.7	0.1	0.1	0.9
4	0.1				0.1	0.8	0.1	0.9
5	0.1				0.1	0.8	0.1	0.9

Fixed interest rate

$$\max NPV = \sum_i \sum_k R_{ik} y_{ik}$$

subject to :

$$\sum_k y_{jk} - a \sum_i \sum_k y_{ik} p(j \mid i,k) \geq \pi_j \qquad \forall j$$

$$y_{ik} \geq 0 \qquad \forall i,k$$

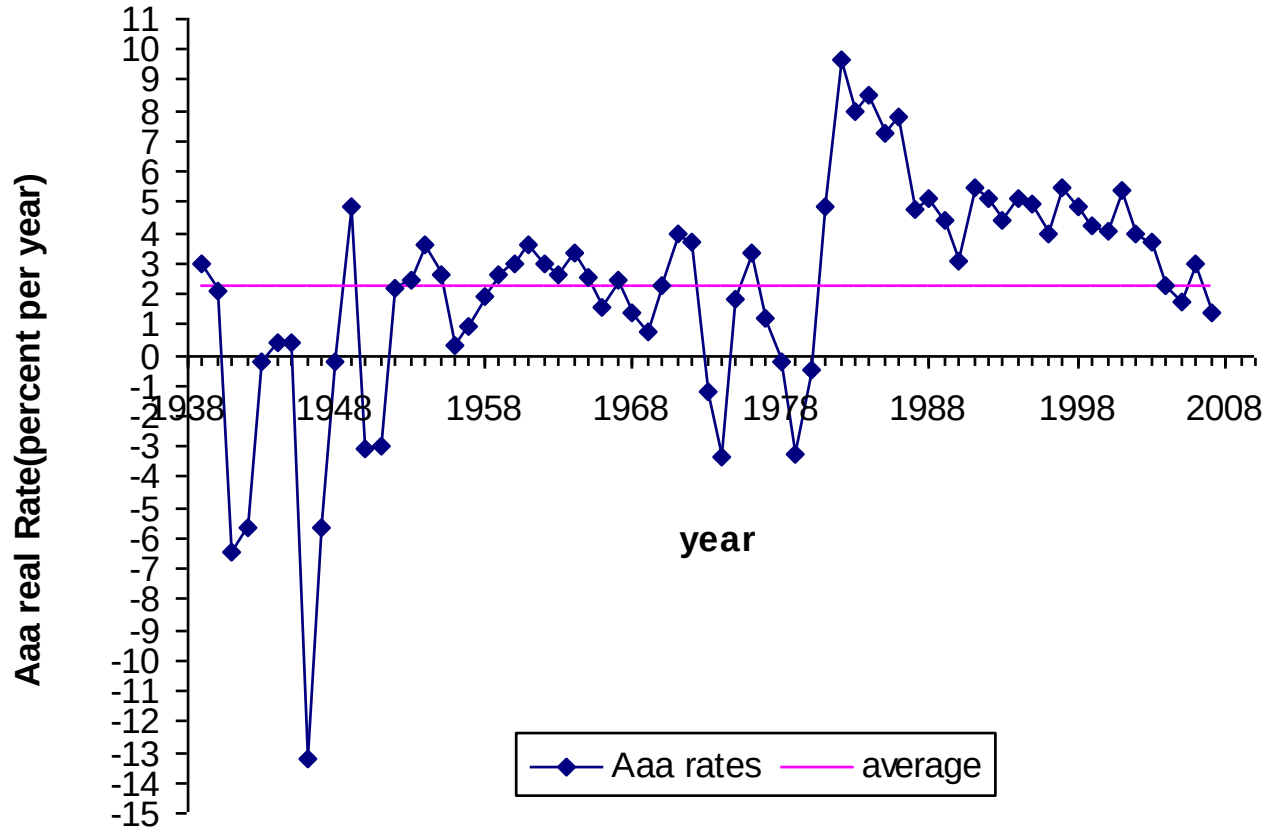
$$X_{ik} = \frac{y_{ik}}{\sum_k y_{ik}}$$

(0,1)

Depends on *i* only

Independent of π_j

Annual rate of return on Aaa bonds



State-dependent interest rate

$$\sum_k y_{jk} - a \sum_i \sum_k y_{ik} p(j \mid i,k) \geq \pi_j \qquad \forall j$$

$$\sum_k y_{jk} - a_j \sum_i \sum_k y_{ik} p(j \mid i,k) \geq \pi_j \qquad \forall j$$

Dual, fixed interest rate

$$\min\ z = \sum_i \pi_i V_i$$

subject to :

$$V_i - a\sum_j p(j \mid i,k) V_j \geq R_{ik} \qquad \forall i,k$$

$$V_i \lessgtr 0 \qquad \forall i$$

Dual, state-dependent interest rate

$$V_i - a\sum_j p(j \mid i,k)V_j \geq R_{ik} \qquad \forall i,k$$

$$V_i - \sum_j a_j\, p(j \mid i,k)V_j \geq R_{ik} \qquad \forall i,k$$

Example

	Do nothing						Harvest & plant		
State	0	1	2	3	4	5	0	1	$/ha
0	1.0						0.1	0.9	-494
1	0.1	0.1	0.7	0.1			0.1	0.9	-117
2	0.1		0.1	0.7	0.1		0.1	0.9	3068
3	0.1			0.1	0.7	0.1	0.1	0.9	6396
4	0.1				0.1	0.8	0.1	0.9	8970
5	0.1				0.1	0.8	0.1	0.9	10790

Stochastic interest rate

	End *j*		
Begin state *i*	Low	High	Interest Rate yr^{-1}
Low	0.70	0.30	0.01
High	0.30	0.70	0.04

Expected=0.025

System states: 6 x 2=12

Transition probabilities without harvest

	L,0	L,1	L,2	L,3	L,4	L,5	H,0	H,1	H,2	H,3	H,4	H,5
L,0	0.70	0	0	0	0	0	0.30	0	0	0	0	0
L,1	0.07	0.07	0.49	0.07	0	0	0.03	0.03	0.21	0.03	0	0
L,2	0.07	0	0.07	0.49	0.07	0	0.03	0	0.03	0.21	0.03	0
L,3	0.07	0	0	0.07	0.49	0.07	0.03	0	0	0.03	0.21	0.03
L,4	0.07	0	0	0	0.07	0.56	0.03	0	0	0	0.03	0.24
L,5	0.07	0	0	0	0.07	0.56	0.03	0	0	0	0.03	0.24
H,0	0.30	0	0	0	0	0	0.70	0	0	0	0	0
H,1	0.03	0.03	0.21	0.03	0	0	0.07	0.07	0.49	0.07	0	0
H,2	0.03	0	0.03	0.21	0.03	0	0.07	0	0.07	0.49	0.07	0
H,3	0.03	0	0	0.03	0.21	0.03	0.07	0	0	0.07	0.49	0.07
H,4	0.03	0	0	0	0.03	0.24	0.07	0	0	0	0.07	0.56
H,5	0.03	0	0	0	0.03	0.24	0.07	0	0	0	0.07	0.56

Transition probabilities with harvest & replant

	L,0	L,1	L,2	L,3	L,4	L,5	H,0	H,1	H,2	H,3	H,4	H,5
L,0	0.07	0.63	0	0	0	0	0.03	0.27	0	0	0	0
L,1	0.07	0.63	0	0	0	0	0.03	0.27	0	0	0	0
L,2	0.07	0.63	0	0	0	0	0.03	0.27	0	0	0	0
L,3	0.07	0.63	0	0	0	0	0.03	0.27	0	0	0	0
L,4	0.07	0.63	0	0	0	0	0.03	0.27	0	0	0	0
L,5	0.07	0.63	0	0	0	0	0.03	0.27	0	0	0	0
H,0	0.03	0.27	0	0	0	0	0.07	0.63	0	0	0	0
H,1	0.03	0.27	0	0	0	0	0.07	0.63	0	0	0	0
H,2	0.03	0.27	0	0	0	0	0.07	0.63	0	0	0	0
H,3	0.03	0.27	0	0	0	0	0.07	0.63	0	0	0	0
H,4	0.03	0.27	0	0	0	0	0.07	0.63	0	0	0	0
H,5	0.03	0.27	0	0	0	0	0.07	0.63	0	0	0	0

Best decision

Interest	Stand state	Fixed interest	Stochastic interest	Gain ($/ha)
Low	0	1	1	73
Low	1	0	0	116
Low	2	1	0	237
Low	3	1	1	73
Low	4	1	1	73
Low	5	1	1	73
High	0	1	1	47
High	1	0	0	63
High	2	1	1	47
High	3	1	1	47
High	4	1	1	47
High	5	1	1	47

Conclusion

- World stochastic, including interest rates
- MDP models powerful
  - General
  - Rigorous
  - Practical
- Decisions differ with interest rate
- Magnitude = f(decision frequency)